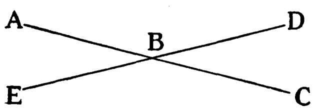
RELATED THEOREMS IN GEOMETRY.

A GREAT deal of confusion prevails among high-school pupils —and is sometimes indicated in text-books—as to the relations between certain important groups of theorems. The distinction between a restatement of an old theorem in new words and a fundamental change in that theorem, requiring new proofs, is seen hazily if at all. For these reasons the following comments on certain important related theorems in geometry may be of value to some other teacher, as their investigation has been to me.

1. Take the proposition, "If two straight lines cross, their vertical angles are equal;" and restate it in terms of the accompanying diagram as follows:



If $\left\{\begin{array}{l}AC \text{ and } ED \text{ are straight lines cross-}\\ \text{ing at } B\end{array}\right.$

Then $\left\{\begin{array}{l}ABD = EBC\\ ABE = DBC\end{array}\right.$

Now state its converse, first in terms of the diagram,

If $\left\{\begin{array}{l}\sphericalangle ABD = \sphericalangle EBC\\ \sphericalangle ABE = \sphericalangle DBC\end{array}\right.$

Then $\left\{\begin{array}{l}AC \text{ and } ED \text{ are straight lines}\\ \text{crossing at } B.\end{array}\right.$

If the attempt is now made to express this in general terms, one is compelled to examine the "If-clauses" in each case very carefully to find out what is essential in them, what is accidental, and what is concealed. It is evident that we have four angles with a common vertex B; also that ABD is adjacent to DBC, DBC to EBC, EBC to ABE, ABE to ABD again; in other words these four angles constitute the total angular magnitude about the point B. The hypothesis of the converse names these angles, in pairs, as equal to each other; how shall we describe them? Opposite is no longer an apt term, as it seems to beg

the question to a certain extent; let us call them non-adjacent. Now the conclusion. We might say "The common side of each pair of angles is in a line with the common side of the other pair."

In general terms, then, the converse of this familiar proposition may be stated as follows:

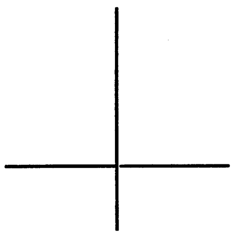
In the figure formed by four angles which occupy the entire angular magnitude about a point in a plane, if the non-adjacent angles are equal, the common side of each pair of adjacent angles is in a line with the common side of the other pair.

By introducing another convenient term for a line extending indefinitely in one direction only,—the ray,—the statement may be more neatly made, as follows:

In the figure formed by four rays from the same point in the same plane, if the non-adjacent angles are equal, the alternate rays are in a line.

In order to see that this is the converse of the original proposition, it is necessary to restate that; and then comparing its new form with the familiar form, it is necessary to see that they are differently worded statements of the same fact. Compare them:

If two straight lines cross each other, the vertical angles are equal



In the figure formed by four rays from the same point in the same plane,—if the alternate rays are in a line, the non-adjacent angles are equal.

Statements which express the same fact in different ways will be called in this article "congruent" statements.

The original form of this proposition, "If two straight lines cross, etc.," has the advantage of presenting the figure from the point of view in which it usually occurs. The recognition of congruent theorems, as in this case, is an important accomplishment for the student.

2. In the amended form, a certain part of the hypothesis is

common to the direct proposition and its converse. So that while the whole hypothesis is comprised in the words, "In the figure formed by four rays from the same point in the same plane, if the non-adjacent angles are equal—," one cannot construct the converse by inverting the hypothesis and conclusion, in accordance with the definition of "converse propositions."

The part of the hypothesis common to both propositions is called the "Universe of Discourse" for those two propositions. The "Universe" of things talked about is narrowed down, first to geometrical figures, then to plane figures, and finally, for these two propositions, to figures formed by four rays from one point.

Similarly, in the well-known propositions about supplementary adjacent angles, the universe of discourse is figures formed by pairs of adjacent angles; the converse propositions being:

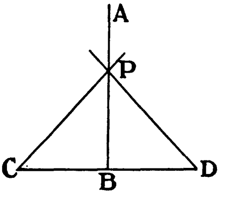
If the exterior sides are in a line
their sum is two right angles.

If their sum is two right angles
their exterior sides are in a line.

This separation of the hypothesis proper from the phrases which define the universe of discourse becomes of great importance in certain locus propositions referred to below.

3. Related propositions can be most accurately studied by separating them into definite statements, each containing, so far as possible, only a single fact or condition. Thus for the proposition

"If one line is perpendicular to another at its middle point, every point in the perpendicular is equally distant from the ends of the line,"—we may construct the following diagrammatic statement:



$AB \perp CD$
P is in AB
$CB = CD$

$PC = PD$

These four clauses may be put, in general terms, as follows:

[X] $AB \perp CD$	A perpendicular is drawn to a line.
[Y] P is in AB	A point lies in the perpendicular.
[Z] $CB = BD$	The perpendicular bisects the line.
[W] $PC = PD$	The point is equally distant from the ends of the line.

For convenience of reference, letter these four statements X, Y, Z and W, respectively.

If we take the first two clauses, X and Y, as universe of discourse for this proposition, the "converse" would be $XYW = Z$; that is, in general terms,—

If equal oblique lines are drawn from any point in the perpendicular to a line, they cut off equal distances from its foot.

Congruent with this are the theorems,—

A perpendicular from the vertex to the base of an isosceles triangle bisects the base.

A diameter perpendicular to a chord bisects it.

Congruent with the original theorem, $XYZ = W$, of which this is the converse, are:

If from any point in the perpendicular to a line, oblique lines are drawn cutting off equal distances from its foot, the oblique lines are equal.

The perpendicular to a chord at its middle point passes through the center of the circle.

Any point in the bisector of a line is equally distant from its ends.

Good teaching for these theorems would consist in showing their congruence with the theorem $XYZ = W$. If a pupil, in "proving" one of them, formally stated his hypothesis and conclusion in terms of a typical diagram, and then referred to $XYZ = W$ as his authority, he would show that he had not received such teaching. A pupil well-taught would reply to his examiner: "This theorem is congruent with [$XYZ = W$] which is proved as follows. . . . "

The pupil then should be trained in recognizing and devising congruent theorems—especially those which introduce compensating negatives, and those which condense by the use of technical terms.

For example, no student should be quit of the theorem $XYZ = W$ till he could recognize it in such forms as:

The bisector of a line contains no point unequally distant from its ends.

4. Most important, among the theorems congruent to a given one, is its "contranominal," or, in the phraseology of Wentworth, the opposite of its converse. The contranominal of any proposition is formed by denying its conclusion for a new hypothesis, and by denying its hypothesis for a new conclusion. The theorem so formed is always congruent with the given one.

Thus, granting the proposition,

If A is B, then C is D, is the same as saying that

If C is not D, then A is not B.

Congruence is more evident in the form, A is not B unless C is D.

It is sometimes easier to prove the contranominal of a proposition; such a proof is called the *reductio ad absurdum*.

It is sometimes more economical to prove the contranominal of a proposition; such a case is the converse, $XYW = Z$, of our original theorem. If we prove instead, that

Of oblique lines drawn, cutting off unequal distances from the foot of a perpendicular, the more remote is the greater,—then we prove $XYW = Z$, and other facts in addition.

The opposite of a theorem is formed by denying the hypothesis for a new hypothesis, and denying the conclusion for a new conclusion. It is evident that the opposite of a theorem is con gruent with its converse.

5. Returning now to the theorem $XYZ = W$, it is evident that we have three converses for it, according to which pair of clauses we take for universe of discourse. In the following diagrams the clauses belonging to the universe of discourse for any set of propositions are in black type.

INVERSES

$X\mathbf{YZ} = W$	$W\mathbf{YZ} = X$
$\mathbf{X}Y\mathbf{Z} = W$	$\mathbf{X}W\mathbf{Z} = Y$
$\mathbf{XY}Z = W$	$\mathbf{XY}W = Z$

For each of these four distinct propositions three different "opposites" may be written, *e. g.:*

$$XYZ = W$$
$$x\mathbf{Y}\mathbf{Z} = w$$[1]
$$\mathbf{X}y\mathbf{Z} = w$$
$$\mathbf{X}\mathbf{Y}z = w$$

The operation of deriving the opposite from any theorem will be called "obversion;" that of deriving the converse, "inversion."

The entire group of propositions derived by simple inversion and obversion from $XYZ = W$ is as follows:[1]

I.	$XYZ = W$	VIII.	$XYw = z$
		IX.	$XyW = z$
		X.	$xYW = z$
II.	$XYW = Z$		
III.	$XWZ = Y$	XI.	$XWz = y$
IV.	$WYZ = X$	XII.	$XwZ = y$
		XIII.	$xWZ = y$
V.	$XYz = w$		
VI.	$XyZ = w$	XIV.	$WYz = x$
VII.	$xYZ = w$	XV.	$WyZ = x$
		XVI.	$wYZ = x$

Of these (VIII) is contranominal to (I) with XY for universe of discourse; (XII) is contranominal to (I) with XZ for universe of discourse; (XVI) is contranominal to (I) with YZ for universe of discourse.

Similarly (II) has for contranominals (V) (XI) (XIV)
" (III) " " (VI) (IX) (XV)
" (IV) " " (VII) (X) (XIII)

So that it is only necessary to prove (I) (II) (III) (IV); the other twelve can be cited in scholia.

On further examination it is evident that in each of the four groups

(I) (VIII) (XII) (XVI); (II) (V) (XI) (XIV);
(III) (VI) (IX) (XV); (IV) (VII) (X) (XIII);

each proposition is contranominal to each of the other three; so that to establish the whole sixteen, it is only necessary to prove one in each group.

[1] In these diagrams the negative of any clause is indicated by writing its representative symbol with a small letter.

It will be found that nearly all elementary text-books of geometry give formal "demonstrations" of (I), (II), (V) and (VI) and also of the three congruent forms of (I) given above, without any comment whatever on the cogency of a proof which quotes its demonstrandum for authority.

The proposition (IV) ($WYZ=X$) may be thus stated:

"The medial of an isosceles triangle is perpendicular to the base." Or,

"The diameter through the middle of a chord is perpendicular to it."

Neither of these propositions is commonly demonstrated.

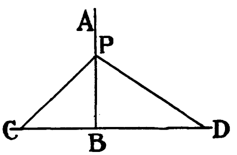
6. Certain facts remain to be emphasized in connection with the theorems of this set that contain negatives.

In discussing the theorem,—

"If from a point in a perpendicular to a line (oblique lines are drawn to the line) XY, the oblique lines that cut off the greater distances are greater than the oblique lines that cut off less"—

$\underline{z}$ or $\bar{z}$
$\underline{w}$ or $\overline{w}$

We may represent the different clauses by letters corresponding to the following diagram:



$AB \perp CD$	X	$\bar{z}$ means $BD<BC$
P is in AB	Y	$\overline{w}$ " $PD<PC$
$BD>CB$	$\underline{z}$	
$PD>PC$	$\underline{w}$	

Geometrical diagrams are types of the class of geometrical figures whose class marks are given by the hypothesis of the proposition which they illustrate. Such classes may be called homogeneous classes, and defined logically by the fact that when any proposition is affirmed of any particular set of members of the class, any other set of members may be substituted . . . that is, any particular set serves as a general representative set.

It often happens that any geometrical figure of a certain type is intimately associated with a particular geometrical figure of another type; *e. g.*, circles and their centers, tangents and their

points of tangency, oblique lines from a point in a perpendicular and the "distances cut off" by them. . . .

Here it is convenient to define conjugate classes as those containing the same number of members, and having each member of one corresponding to a particular member of the other.

Now we can state a theorem which will be useful in discussing the proposition $XY\underline{z}=\underline{w}$:

Theorem. If, as hypothesis, any relation is affirmed among members of a homogeneous class, and, as conclusion, a relation is established among the corresponding members of a conjugate class, then the terms of the hypothesis may be transposed in any order, and the corresponding relation in the conjugate class is a valid inference.

The condition $\underline{z}$ ($BD>CB$) does not logically differ from $\bar{z}$ ($CB>BD$) because the geometrical figures CB and BD are each typical diagrams, members of a homogeneous class; similarly for $\underline{w}$ and $\bar{w}$, considered by itself. But PD and BD are corresponding types of conjugate classes—so are PC and CB—and therefore the combined statements $\underline{z}=\underline{w}$ imply $\bar{z}=\bar{w}$, without implying $\bar{z}=\underline{w}$ or $\underline{z}=\bar{w}$. That is the new fact which is established by the proof of this after $XYW=Z$ has been proven.

It is more economical to prove this proposition, and from it infer proposition $XYW=Z$. Thus for establishing the whole series of sixteen propositions we prove the following four:

If from a point in a perpendicular to a line two oblique lines are drawn to the line, cutting off unequal distances from the foot of the perpendicular, the one that cuts off the greater distance is the greater; and conversely, if two unequal oblique lines are drawn, the greater of them cuts off a greater distance (from the foot of the perpendicular) than the other does.

V	VIII
$XY\underline{z}=\underline{w}$	$XY\underline{w}=\underline{z}$

If a perpendicular is erected at the middle of a straight line, any point not in the perpendicular is unequally distant from the extremities of the line.

VI

$XyZ = w$

The medial of an isosceles triangle is perpendicular to the base.

IV

$XWZ = Y$

There is in fact a much larger series of propositions connected with $XYWZ$. These are sometimes stated in some such form as this :

X	$AB \perp CD$
Y	P in AB
Z	$CB = BD$
W	$PC = PD$
U	$BCP = BDP$
V	$BPC = BPD$

If any three of these propositions are true, all are true.

In order to establish this comprehensive statement it is necessary to prove 18 propositions, namely : four each for the 16-set $XYZ = W$, the 16-set $XYZ = U$, and the 16-set $XYZ = V$; also six for propositions containing U and V together in the hypothesis.

If two of the six statements $XYZWUV$ were affirmed, and another denied, the other three would be denied implicitly ; for not one of them could be true without making three propositions true out of the series, and thus establishing all.

Thus if two of the statements were true the other four must be either all true or all false.

If one statement is true and another false, one of the remaining four, and one only, may be true ; and then the rest must be false ; or else all the remaining four must be false.

If one statement is true, all the others may be false, or all false but one, or all true.

If there are, in general, $\underline{n}$ statements, such that the affirmation of any $\underline{k}$ of them implies the affirmation of all the rest, then —

In order to infer the denial of any particular one of these statements it is necessary to affirm $k-1$ and deny one ; then all the rest would be denied implicitly.

[For if another one should be true, k statements would be affirmed, which would imply all the rest.]

In order to a valid inference, k statements must each be either affirmed or denied.

If one statement is false, not more than $k-1$ can be true.

If $n-k$ statements are denied at least one other is; if j statements are denied, $n-k+1-j$ others are.

The total number of statements denied must be either $=o$ or $> n-k$.

GEO. W. EVANS

ENGLISH HIGH SCHOOL,
Boston, Mass.